MODELING FLOW AND MASS TRANSFER IN CYLINDRICAL POROUS MEDIA FILTERS

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The study addresses fluid filtration processes in soil and filter systems, focusing on their significance for land reclamation, hydrology, and hydrogeology, particularly in Uzbekistan. A mathematical model of unsteady filtration in sandwich-type reservoirs was developed using parabolic partial differential equations with boundary conditions, and an analytical solution was obtained via the Laplace transform. Computational experiments revealed that pressure in both layers grows exponentially, with water exchange largely controlled by the piezo conductivity of the highly permeable layer and the filtration coefficient of the weakly permeable one. Additionally, a numerical algorithm was proposed to study liquid and ionized solution filtration, accounting for suspension concentration changes, sediment and gel deposition, pressure growth in the column, and physical properties of the fluid. Simulation results, presented graphically, support optimization of filtration unit operation under varying suspension characteristics and external factors.[1-2]

The study, along with formulas for determining fluid flow in a well, examines the laws of pressure distribution regulation under direct radial flow conditions, taking into account horizontal and vertical directions.[3]

This work presents a unified mathematical model of processes in a cylindrical porous filter with axial symmetry. The model combines fluid flow and pressure distribution, multicomponent convective-diffusive transport, adsorption with LDF kinetics and Langmuir isotherm, clogging due to particle deposition, and membrane interactions. Hydraulic behavior is described by Brinkman–Darcy equations, permeability–porosity dependence by the Kozeny–Carman relation, and solute transport by ADR equations. Clogging is modeled through porosity reduction over time, while membrane surface massing and permeation are set via boundary conditions.

Below are the equations for a cylindrical porous filter.

Equations for hydraulics (Brinkman-Darcy) (1) - (4):

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_z}{\partial z}=0. \quad (1)$$

$$-\frac{\mu}{\kappa(\varepsilon)}u_r+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right)-\frac{\partial p}{\partial r}=0. \quad (2)$$

$$-\frac{\mu}{\kappa(\varepsilon)}u_z+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{\partial^2 u_z}{\partial z^2}\right)-\frac{\partial p}{\partial z}=0. \quad (3)$$

$$\kappa(\varepsilon)=\frac{\varepsilon^3\ d_f^2}{180(1-\varepsilon)^2}. \quad (4)$$

Where: r – radial coordinate[m], distance from the center to the wall, z – vertical coordinate [m], direction from top to bottom, $u_r=u_r(r,z,t)$ – radial velocity component [m/s], $u_z=u_z(r,z,t)$ – vertikal velocity component [m/s], θ – angular coordinate [rad], not taken when the axis is symmetrical, $u_\theta(r,z,t)$ – angular velocity component [m/s], zero due to symmetry, μ – dynamic viscosity [Pa·s], for water~0.001 (20 °C da), $\kappa(\varepsilon)$ – permeability [m²], flow capacity, $p=p(r,z,t)$ – pressure [Pa], $\varepsilon(r,z,t)$ – porosity, $(0<\varepsilon<1)$, proportion of free space, d_f - filter particle diameter [m].

For substance transport (for each $i=1,\ldots,N$), the following equations (5) – (8) are expressed.

$$\frac{\partial(\varepsilon c_i)}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rc_iu_r)+\frac{\partial}{\partial z}(c_iu_z)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\varepsilon D_{i,eff}\frac{\partial c_i}{\partial r}\right)+$$

$$+\frac{\partial}{\partial z}\left(\varepsilon D_{i,\,eff}\frac{\partial c_i}{\partial z}\right)-r_{i,\,bulk}-a_s r_{i,\,surf}. \quad (5)$$

$$\frac{\partial q_i}{\partial t}=k_{f,i}\left(q_i^*(\mathbf{c})-q_i\right). \quad (6)$$

$$q_i^*(c)=\frac{q_{i,\max}b_ic_i}{1+\sum_{j=1}^{N}b_jc_j}. \quad (7)$$

$$r_{i,\,surf}=\rho_s\frac{\partial q_i}{\partial t}. \quad (8)$$

Where: t – time [s], for the dynamics of the process, , N – number of components, $c_i(r,z,t)$ – concentration of the i-th component in water [mol/m³ or kg/m³], $D_{i,\text{eff}}$ – effective diffusion coefficient [m²/s], $r_{i,\text{bulk}}$ – bulk reaction rate in solution [mol/(m³·s)], a_s – surpace area of per unit volume [m²/m³], $r_{i,\text{surf}}$ – surface loss rate [mol/(m²·s)], $q_i(r,z,t)$ – capture quantity in the solid phase [mol/kg], $k_{f,i}$ – LDF(linear driving force) kinetic coefficient [1/s], $q_i^*(\mathbf{c})$ – equilibrium adsorption quantity [mol/kg], $q_{i,\max}$ – maximum adsorption capacity [mol/kg], b_i – Langmuir affinity coefficient [m³/mol], ρ_s – density of adsorbent particles [kg/m³].

The colmatation (optional, particle) equations (9)-(10) defined as follows.

$$\frac{\partial(\varepsilon c_p)}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\, c_p u_r\right)+\frac{\partial}{\partial z}\left(c_p u_z\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\, \varepsilon D_p \frac{\partial c_p}{\partial r}\right)+ \\ +\frac{\partial}{\partial z}\left(\varepsilon D_p \frac{\partial c_p}{\partial z}\right)-k_{\text{dep}}\, c_p, \tag{9}$$

$$\frac{\partial \varepsilon}{\partial t}=-\frac{\nu_{\text{dep}}}{\rho_{\text{dep}}} k_{\text{dep}}\, c_p. \tag{10}$$

Where: $c_p(r,z,t)$ – concentration of particulate matter in water [kg/m³], D_p – diffucion coefficient of particles [m²/s], k_{dep} – depolation coefficient [1/s], ν_{dep} – volume fraction of depolation [–], ρ_{dep} – density of settled particles [kg/m³].

Membrane equations (if available below, $G_m \equiv G_{out}$) are:

$$J_v = L_p\left(\Delta P-\sum_{i=1}^{N}\sigma_i\, \Delta\pi_i\right), \tag{11}$$

$$J_i=(1-\sigma_i)c_{i,\text{avg}}\, J_v - P_{s,i}\,\Delta c_i, \tag{12}$$

$$-\varepsilon D_{i,\,eff}\frac{\partial c_i}{\partial n}+c_i\, J_v = J_i, u_n|_{G_m}=J_v, \tag{13}$$

$$J_v=\frac{\Delta P}{\mu(R_m+R_c)}, R_c=\alpha_c\,\delta_c, \frac{\partial \delta_c}{\partial t}=\frac{J_v\, c_{p,m}}{\rho_c}. \tag{14}$$

Where: G_{in} – inlet surface (from above, $z=0$), G_{out} – outlet surface (from below, $z=H$), J_v – volumetric flow through the membrane (permeat flux) [m/s], L_p – membrane hydraulic permeability [m/(Pa·s)], ΔP – transmembrane pressure difference [Pa], σ_i – reflection coefficient (0–1), π_i – osmotic pressure [Pa], J_i – i-membrane flow of the i-th component [mol/(m²·s)], $c_{i,\text{avg}}$ – average concentration near the membrane [mol/m³], $P_{s,i}$ – salt permeability [m/s], $\partial/\partial n$ – normal derivative operator (at the membrane level), R_m – internal membrane resistance [1/m], R_c – cake resistance [1/m], α_c – specific cake resistance [m/kg], $\delta_c(r,z,t)$ – cake thickness [m], $c_{p,m}$ – particle concentration on the membrane [kg/m³], ρ_c – cake density [kg/m³].

Boundary conditions are given in equations (15)-(19):

Entrance (from top, G_{in}) boundary conditions

$$p=p_{in}, c_i=c_{i,in}(t)(i=1,\ldots,N). \tag{15}$$

Exit (from below, G_{out}, $z=H$) boundary conditions

$$p=p_{out}\approx p_{atm}, -\varepsilon D_{i,\,eff}\frac{\partial c_i}{\partial z}+c_i\, u_z=0. \tag{16}$$

Boundary conditions of the side wall (G_r, $r = R$)

$$u_r = 0, \frac{\partial u_z}{\partial r} = 0, \frac{\partial c_i}{\partial r} = 0. \quad (17)$$

Boundary conditions for the axis of symmetry (G_0, $r = 0$)

$$u_r = 0, \frac{\partial u_z}{\partial r} = 0, \frac{\partial c_i}{\partial r} = 0. \quad (18)$$

Membrane level (if applicable, G_m) boundary conditions

$$u_n = J_v, -\varepsilon D_{i,\,eff} \frac{\partial c_i}{\partial n} + c_i J_v = J_i. \quad (19)$$

The initial conditions (in the case $t = 0$) are determined using equations(20)-(23)

$$u_r(r,z,0) = u_{r,0}(r,z), u_z(r,z,0) = u_{z,0}(r,z), p(r,z,0) = p_0(r,z), \quad (20)$$

$$c_i(r,z,0) = c_{i,0}(r,z), q_i(r,z,0) = q_{i,0}(r,z)(i = 1,\ldots,N), \quad (21)$$

$$\varepsilon(r,z,0) = \varepsilon_0(r,z), \delta_c(r,z,0) = 0. \quad (22)$$

When solving the above equations, exact analytical solutions lead to complex calculations and difficulties. Therefore, the results of numerical methods are considered tools for solving these equations. One of the most effective methods for applying numerical solutions is the finite difference method. Determining the solution in this way involves the following steps:

References

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